Correlation coefficients between the velocity difference and local average dissipation of turbulence

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In order to assess the refined similarity hypothesis (RSH) of the Kolmogorov 1962 (K62) theory, recently many authors have measured the correlation coefficients between $|\Delta u_r|$ (or Δu_r) and ϵ_r [or $(r\epsilon_r)^{1/3}$] of the high-Reynolds-number turbulence, here Δu_r is the velocity difference across a distance r, and ϵ_r is the local average dissipation over the scale r which is represented by its one-dimensional (1D) surrogate in experiments. We study how the correlation coefficients change with r in the inertial range according to the K62 theory, using three typical models of intermittency. It is found that the experimental data of correlation coefficients contradict consequences of the K62 theory no matter which of the three intermittency models is used. This finding may imply that the stochastic variable $V=\Delta u_r/(r\epsilon_r)^{1/3}$ depends on $(r\epsilon_r)^{1/3}$ in the inertial range captured in experiments while ϵ_r is represented by its 1D surrogate, and highlights the issue on the RSH of the K62 theory. [S1063-651X(96)03507-6]

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We study the inertial-range statistics of turbulence. Let Δu_r be the velocity difference across a distant r, ϵ_r be the local average dissipation over the scale r, $\eta_r=(\nu^3/\epsilon_r)^{1/4}$ be the internal scale, and ν be the kinematic viscosity. According to the refined similarity hypotheses (RSH) of the Kolmogorov 1962 (K62) theory [1], in the inertial range,

$$V=\Delta u_r/(r\epsilon_r)^{1/3} \tag{1}$$

is an universal stochastic variable independent of $(r\epsilon_r)^{1/3}$, hence

$$\langle|\Delta u_r|^q\rangle=\langle|V|^q\rangle r^{q/3}\langle\epsilon_r{}^{q/3}\rangle. \tag{2}$$

Here $\langle\,\rangle$ denotes the ensemble average. Theoretical, experimental, and numerical studies of turbulent intermittency establish the following scaling law in the inertial range:

$$\langle\epsilon_r^n\rangle/\langle\epsilon\rangle^n=(r/L)^{-\mu_n}. \tag{3}$$

Here $\langle\epsilon\rangle=\langle\epsilon_r\rangle$ is independent of r, L is a macroscale depending on the macrostructure of the turbulence, and μ_n is the intermittency exponent of order n. In K62 theory, the inertial range is defined as $L\gg r\gg\eta$, where η is the upper limit (excluding cases of negligibly small probability) of η_r [1]. Different intermittency models predict different expressions for μ_n. For example, the log normal model gives [1,2]

$$\mu_n=\mu n(n-1)/2, \tag{4a}$$

the multifractal p model gives [3]

$$\mu_n=(n-1)+\log_2[p^n+(1-p)^n], \tag{4b}$$

and the She-Leveque model gives [4]

$$\mu_n=2n/3-2[1-(2/3)^n]. \tag{4c}$$

The log normal model (4a) agrees well with the experimental data of low-order moments when $\mu=0.15-0.25$ [5], and the p model (4b) agrees well with experiments for generalized dimensions of the dissipation field when $p=0.7$ [3]. We note that the L in (3) differs from Kolmogorov's external scale by a numerical constant so that the scaling law is expressed by an equality.

Recently, triggered by the paper of Hosokawa and Yamamoto [6], many authors [7–12] have made experimental and numerical studies to assess the RSH of the K62 theory. One of the statistical quantities studied by them is the correlation coefficient between $|\Delta u_r|$ and ϵ_r or $(r\epsilon_r)^{1/3}$ of high-Reynolds-number turbulence. Praskovsky [8] and Zhu, Antonia, and Hosokawa [12] measured the correlation coefficient between $|\Delta u_r|$ and ϵ_r defined as (they use σ to denote the root-mean-square value instead of the standard deviation),

$$\rho_1=\langle(|\Delta u_r|-\langle|\Delta u_r|\rangle)(\epsilon_r-\langle\epsilon_r\rangle)\rangle/(\langle\Delta u_r^2\rangle\langle\epsilon_r^2\rangle)^{1/2}. \tag{5}$$

Stolovitzky, Kailasnath, and Sreenivasan [7] and Thoroddsen and Van Atta [9] measured the correlation coefficient between $|\Delta u_r|$ and $(r\epsilon_r)^{1/3}$ defined as

$$\rho_2=\langle(X-\langle X\rangle)(Y-\langle Y\rangle)\rangle/[\langle(X-\langle X\rangle)^2\rangle \times\langle(Y-\langle Y\rangle)^2\rangle]^{1/2},\quad X=|\Delta u_r|,\quad Y=(r\epsilon_r)^{1/3}. \tag{6}$$

In these experiments, ϵ_r is represented by its one-dimensional (1D) surrogate, and it is believed that the large experimental values of ρ_1 and ρ_2 obtained by these authors represent some experimental evidence supporting RSH of the K62 theory. However, without knowing how the correlation coefficients change with r in the inertial range according to K62 theory, it is hardly possible to judge whether the experimental values of ρ_1 and ρ_2 support K62 theory or not.

The outline of this paper is as follows. We study how the correlation coefficients ρ_1 and ρ_2 change with r in the inertial range according to the K62 theory, and then it is shown that their experimental data contradict consequences of K62 theory no matter which of the three models (4a)–(4c) is used for the intermittency exponent μ_n, hence highlights the issue on RSH of K62 theory. Finally we discuss the experimental

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TABLE I. α_1, α_2, F_m, X_m, and β for three typical intermittency models.

Model	α_1	α_2	F_m	X_m	β
Log normal (μ=0.2)	0.0444	0.0667	0.326	1.07*E*-6	0.0222
p-model (*p*=0.7)	0.0424	0.0788	0.369	1.61*E*-6	0.0281
She-Leveque	0.0428	0.0843	0.382	2.47*E*-6	0.0319

results by Thoroddsen [13] using different 1D forms of the local average dissipation and relevant problems.

Firstly we study the correlation coefficient ρ_1 defined by (5). According to the RSH of the K62 theory, V is independent of $(r\epsilon_r)^{1/3}$ in the inertial range, from (1) and (5) we have

$$\rho_1=(\langle|V|\rangle/\langle V^2\rangle^{1/2})[\langle\epsilon_r^{4/3}\rangle/(\langle\epsilon_r^{2/3}\rangle\langle\epsilon_r^2\rangle)^{1/2}]$$
$$\times[1-\langle\epsilon_r^{1/3}\rangle\langle\epsilon_r\rangle/\langle\epsilon_r^{4/3}\rangle],$$

then, using (3), we finally obtain

$$\rho_1/\gamma=(r/L)^{\alpha_1}[1-(r/L)^{\alpha_2}], \tag{7a}$$

$$\gamma=\langle|V|\rangle/\langle V^2\rangle^{1/2}, \tag{7b}$$

$$\alpha_1=(\mu_{2/3}+\mu_2)/2-\mu_{4/3}, \quad \alpha_2=\mu_{4/3}-\mu_{1/3}-\mu_1. \tag{7c}$$

According to K62 theory, in the inertial range, γ is a universal constant independent of r. Both α_1 and α_2 are positive, their values are given in Table I for three typical intermittency models. Figure 1 shows a plot of (7) over a very wide inertial range, ρ_1/γ approaches zero as r/L decreases to zero. Therefore, according to K62 theory, the correlation coefficient ρ_1 does depend on r within the inertial range. This result denies Praskovsky's conjecture [8] that ρ_1 does not depend on r within the inertial range according to K62 theory. Moreover, ρ_1/γ attains its maximum value F_m at $r/L=X_m$ (F_m and X_m are given in Table I), ρ_1 increases with r in the range $(r/L)<X_m$, and decreases in the range $(r/L)>X_m$. The inertial-range r should be less than the macroscale L, so the part of Fig. 1 near $r/L=1$ is not observable. It is hardly possible for today's experiments to observe an inertial range as wide as shown in Fig. 1 (and Fig. 4 below), of course some part of Fig. 1 (and Fig. 4) should be observed by today's experiments of high-Reynolds-number turbulence if the K62 theory is valid.

In order to compare experimental data with the consequence (7) of the K62 theory, it is necessary to estimate the value of $\gamma=\langle|V|\rangle/\langle V^2\rangle^{1/2}$. In the inertial range of the high-Reynolds-number turbulence, the *pdf* of V is close to a Gaussian *pdf* [7,11,14]. For a Gaussian *pdf*, $\langle|V|\rangle/\langle V^2\rangle^{1/2}=0.798$. In fact, as shown in Fig. 2 using a simple *pdf* model, $\langle|V|\rangle/\langle V^2\rangle^{1/2}$ changes slightly around 0.8 as V deviates from Gaussianity. Therefore it is reasonable to adopt $\gamma=0.8$. Figure 3 shows a comparison of the plot of (7) with the experimental data of Praskovsky [8] and Zhu, Antonia, and Hosokawa [12]. The three sets of experimental data of ρ_1 in the inertial range decrease as r increases, so Fig. 3 does not include the range $(r/L)<X_m$, within which ρ_1 increases with r. The value of L in (3) and (7) depends on the macrostructure of turbulence and may be unknown, moreover different authors have used different normalization scales to plot their experimental data. Changing the normalization scale is equivalent to translating all data points of the same turbulent flow as a solid body horizontally. Therefore, we adopt the ''translation criterion'' to compare experimental data with (7), i.e., we move horizontally all data points of the same turbulent flow as a solid body in Fig. 3 to discover

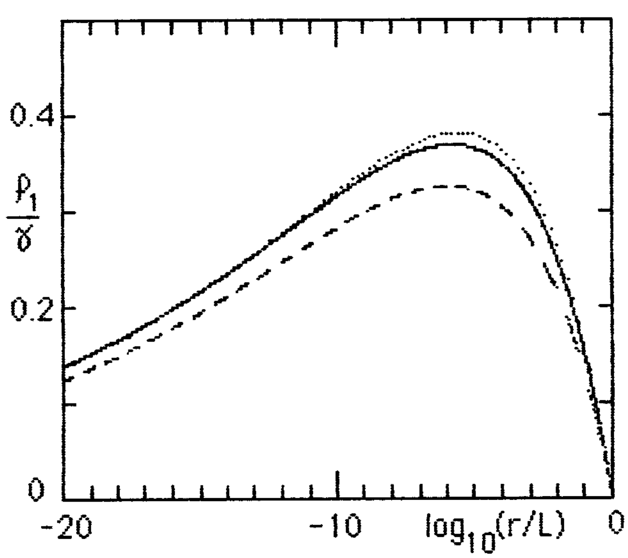

FIG. 1. ρ_1/γ vs $\log_{10}(r/L)$ by (7) for three typical intermittency models. --- , log normal model; — , multifractal p model; $\cdots\cdot$, She-Leveque model.

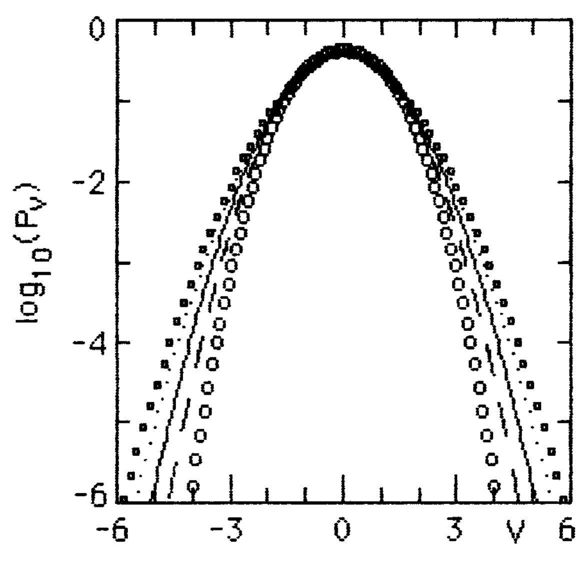

FIG. 2. $\text{Log}_{10}(P_v)$ vs V. P_v is *pdf* of V. — Gaussian *pdf*, $\gamma=0.798$; ■, $\gamma=0.79$; $\cdots\cdot$, $\gamma=0.793$; --- , $\gamma=0.803$; ∘∘∘ , $\gamma=0.81$; $\gamma=\langle|V|\rangle/\langle V2\rangle^{1/2}$.

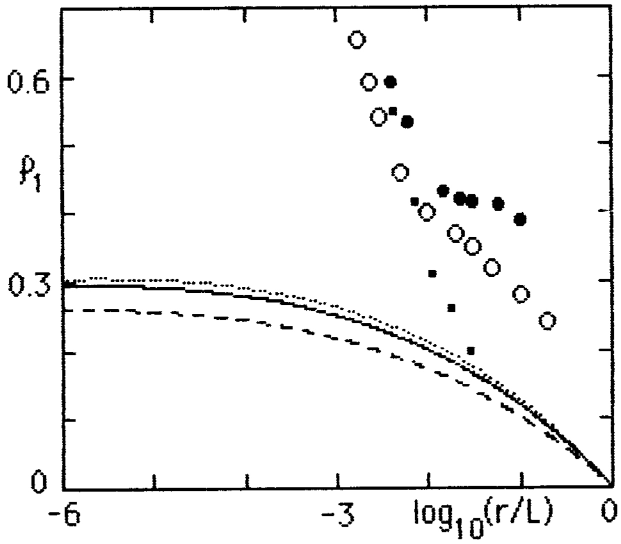

FIG. 3. ρ_1 vs $\log_{10}(r/L)$. γ=0.8. ○, Praskovsky; ●, Zhu, Antonia, and Hosokawa (ASL); ■, Zhu, Antonia, and Hosokawa (jet); all experimental data are within inertial range. ---, (7) and log normal model; —, (7) and multifractal p model; $\cdots\cdot$, (7) and She-Leveque model.

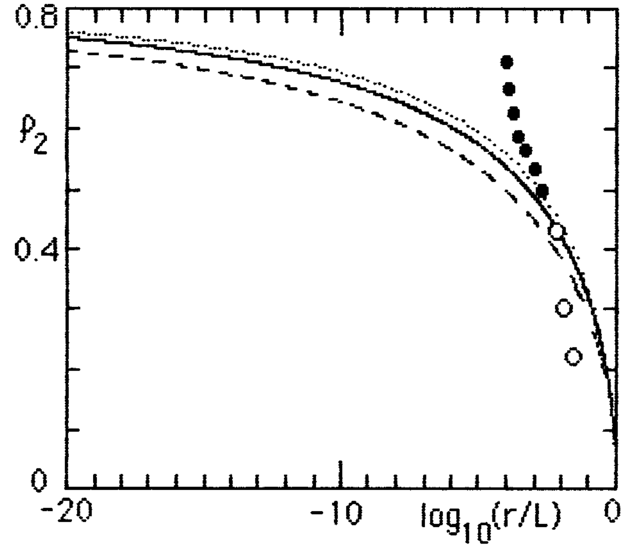

FIG. 4. ρ_2 vs $\log_{10}(r/L)$. γ=0.8. ●, Stolovitzky *et al.*; ○, Thoroddsen and Van Atta; all experimental data are within inertial range. ---, (8) and log normal model; —, (8) and multifractal p model; $\cdots\cdot$, (8) and She-Leveque model.

possible agreement between experimental data and the plot of (7). If we fail to discover agreement by the translation, we conclude that the experimental data contradict (7). Of course, the ''translation criterion'' is only a necessary condition, and not sufficient for the agreement between the experimental data and the plot of (7). We have taken advantage of such a translation of data points to avoid an overlap of data points of different turbulent flows in Fig. 3. Figure 3 clearly shows that the experimental data of the inertial-range ρ_1, reported by Praskovsky [8] and Zhu, Antonia, and Hosokawa [12] contradict the consequence (7) of the K62 theory no matter which of the three typical intermittency models (4a)–(4b) is used. In particular, most experimental values of the inertial-range ρ_1 are much larger than γF_m, which is the upper limit of the inertial-range ρ_1 according to (7) and ranges from 0.26 to 0.3 for three typical intermittency models. Therefore, contradicting general belief, large values of ρ_1 are not an experimental evidence supporting K62 theory. Since γ cannot exceed 1 by (7b), the disagreement between (7) and experimental data remains unchanged when γ takes values other than 0.8 or even changes with r. We have used the log normal model (4a) with different values of μ between 0.1 and 0.3 to plot (7) and compared them with the experimental data (but not shown here), and it is found that the disagreement between (7) and the experimental data does not depend on which value of μ is used.

Now we study whether the experimental data of ρ_2 in the inertial range are compatible with K62 theory. According to the RSH of the K62 theory, V is independent of $(r\epsilon_r)^{1/3}$ in the inertial range, from (1), (6), and (7b) we have

$$\rho_2=\gamma[(1-\langle\epsilon_r^{1/3}\rangle^2/\langle\epsilon_r^{2/3}\rangle)/(1-\gamma^2\langle\epsilon_r^{1/3}\rangle^2/\langle\epsilon_r^{2/3}\rangle)]^{1/2},$$

then, using (3), finally we obtain

$$\rho_2/\gamma=\{[1-(r/L)^{\beta}]/[1-\gamma^2(r/L)^{\beta}]\}^{1/2}, \tag{8a}$$

$$\beta=\mu_{2/3}-2\mu_{1/3}. \tag{8b}$$

The values of β for three typical intermittency models are given in Table I. Figure 4 shows the plot of (8) and a comparison of (8) with the experimental data of Stolovitzky, Kailasnath, and Sreenivasan [7] and Thoroddsen and Van Atta [9]. By (8), ρ_2/γ approaches 1 as r/L decrease to zero. The inertial-range r should be less than L, so the part of the plot of (8) near $r/L=1$ is not observable. Although their magnitude are compatible with (8), the two sets of experimental data of the inertial-range ρ_2 change much faster than the plot of (8). By the ''translation criterion'' mentioned above, Fig. 4 clearly indicates that the experimental data of ρ_2 in the inertial range contradict the consequence (8) of K62 theory no matter which of the three typical intermittency models (4a)–(4c) is used.

The experimental data of the correlation coefficients reported in Refs. [7, 8, 9, and 12] has been regarded by their authors as some experimental evidence supporting the RSH based upon the assumption that ϵ_r can be represented by its 1D surrogate. Hence we also adopt the same assumption here to discuss whether these experimental data actually support RSH. Formulas (7) and (8) for the correlation coefficients ρ_1 and ρ_2 are logical consequences of the following two ingredients of the K62 theory: one is that $V=\Delta u_r/(r\epsilon_r)^{1/3}$ is a universal stochastic variable independent of $(r\epsilon_r)^{1/3}$ in the inertial range (RSH), and another is the inertial-range scaling law (3) of turbulent fluctuations. At present, we have no doubt about the scaling law (3). In the plotting of Figs. 3 and 4, three typical intermittency models (4) are used to evaluate low-order intermittency exponents $\mu_n(n\leqslant 2)$, and the difference in the three corresponding theoretical curves is much less than the disagreement between experimental data and the consequences (7) and (8) of the K62 theory. Therefore, in this author's opinion, the disagreement between experimental data and consequences of K62 theory, clearly shown in Figs. 3 and 4, seems to indicate that $V=\Delta u_r/(r\epsilon_r)^{1/3}$ depends on $(r\epsilon_r)^{1/3}$ in the inertial range captured in the experiments while ϵ_r is represented by its 1D surrogate. Another opinion is that this disagreement may reflect the problem with the intermittency models (4) and even the scaling law (3). The dependence of V upon $(r\epsilon_r)^{1/3}$ may have different effects on different statistical properties of inertial range. In some cases, this effect may be small, so it is reasonable to assume that V is approximately independent of $(r\epsilon_r)^{1/3}$ in the inertial range. The dependence of V upon $(r\epsilon_r)^{1/3}$ has a significant effect upon the correlation coefficients between velocity difference and local average dissipation. This particular behavior of the correlation coefficients provides a convenient quantitative way to study whether $V=\Delta u_r/(r\epsilon_r)^{1/3}$ depends on $(r\epsilon_r)^{1/3}$ in the inertial range. It should be noted that over the whole small-scale range V may depend on r, another factor than $(r\epsilon_r)^{1/3}$. According to K62 theory, V is independent of r in the inertial range $L \gg r \gg \eta$, where η is the upper limit (excluding cases of negligibly small probability) of the internal scale η_r [1].

A few words should be said about two other correlation coefficients between the velocity difference and the local average dissipation, which are obtained from (5) and (6) by replacing the modulus $|\Delta u_r|$ with Δu_r, that is,

$$\rho_3=\langle(\Delta u_r-\langle\Delta u_r\rangle)(\epsilon_r-\langle\epsilon_r\rangle)\rangle/(\langle\Delta u_r^2\rangle\langle\epsilon_r^2\rangle)^{1/2}, \quad (9)$$

$$\rho_4=\langle(X-\langle X\rangle)(Y-\langle Y\rangle)\rangle/[\langle(X-\langle X\rangle)^2\rangle \times\langle(Y-\langle Y\rangle)^2\rangle]^{1/2}, \quad X=\Delta u_r, \quad Y=(r\epsilon_r)^{1/3}. \quad (10)$$

The experimental values of ρ_3 and ρ_4 in the inertial range, reported by Zhu, Antonia, and Hosokawa [12] and Stolovitzky, Kailasnath, and Sreenivasan [7] are positive and about the order of 0.1. It is easy to prove that, in the inertial range, ρ_3 and ρ_4 should be zero if V is a universal stochastic variable independent of $(r\epsilon_r)^{1/3}$ (RSH). Experimental data of a zero quantity might deviate from zero due to measurement errors, and generally the deviations scatter irregularly around zero. Hence, the regular positive values of ρ_3 and ρ_4 in the inertial range, reported by Stolovitzky, Kailasnath, and Sreenivasan [7] and Zhu, Antonia, and Hosokawa [12], seem to indicate that V is not a universal stochastic variable independent of $(r\epsilon_r)^{1/3}$ in the inertial range captured in their experiments while ϵ_r is represented by its 1D surrogate. Sreenivasan pointed out that it may related to other issues such as the finite-Reynolds-number effect and so forth, which are not known.

Kolmogorov [1] gave two different formulations of his RSH. In the first formulation $U_r=(r\epsilon_r)^{1/3}$ is used as a normalization velocity to get the dimensionless stochastic variable $V=\Delta u_r/U_r$, and ϵ_r is represented by a special form suggested by Oboukhov (the dissipation rate averaged over a sphere of scale r). One might ask whether Oboukhov's form of ϵ_r is indispensable for the RSH of the K62 theory or just one of many possible candidates. In the second formulation of his RSH, Kolmogorov tried to free from the special selection of ϵ_r assuming Oboukhov's form. Let (x,y,z) be the coordinates and (u,v,w) be the velocity. In almost all experiments, ϵ_r is represented by the 1D surrogate $\epsilon_r[u/x]$ which is $15\nu(\partial u/\partial x)^2$ averaged over an interval of scale r. Thoroddsen [13] used $\epsilon_r[w/x]$ and $\epsilon_r[u/z]$, corresponding to $7.5\nu(\partial w/\partial x)^2$ and $7.5\nu(\partial u/\partial z)^2$, respectively, and found that if ϵ_r is represented by $\epsilon_r[w/x]$ or $\epsilon_r[u/z]$ the experimental data of the conditional average $\langle|\Delta u_r||\epsilon_r\rangle$ contradict RSH. However, if ϵ_r is represented by $\epsilon_r[u/x]$, the experimental data of $\langle|\Delta u_r||\epsilon_r\rangle$ support RSH. Chen *et al.* [10] obtained the same results by numerical investigations and reported that the behavior of the conditional average $\langle|\Delta u_r||\epsilon_r\rangle$, with ϵ_r being Oboukhov's form, is similar to $\langle|\Delta u_r||\epsilon_r\rangle$ with ϵ_r being the 1D surrogate $\epsilon_r[u/x]$. Therefore $\epsilon_r[w/x]$ or $\epsilon_r[u/z]$ cannot be used as the 1D surrogate of Oboukhov's ϵ_r in the matter of RSH. In this paper the 1D surrogate of ϵ_r means $\epsilon_r[u/x]$, which is in most experiments. Although there is some experimental evidence supporting the RSH while ϵ_r is represented by its 1D surrogate $\epsilon_r[u/x]$, this paper argues that the experimental data of correlation coefficients contradict the RSH while ϵ_r is represented by its 1D surrogate $\epsilon_r[u/x]$, hence highlights the issue on the RSH of K62 theory. For the present, it is an open problem whether the experimental correlation coefficients also contradict the RSH while ϵ_r is Oboukhov's form.

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